Factors of binomial sums from the Catalan triangle

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Abstract. By using the Newton interpolation formula, we generalize the recent identities on the Catalan triangle obtained by Miana and Romero as well as those of Chen and Chu. We further study divisibility properties of sums of products of binomial coefficients and an odd power of a natural number. For example, we prove that for all positive integers \(n_1, \ldots, n_m, n_{m+1} = n_1\), and any nonnegative integer \(r\), the expression

\[ n_1^{-1} \binom{n_1 + n_m}{n_1} \sum_{k=1}^{n_1} k^{2r+1} \prod_{i=1}^{m} \binom{n_i + n_{i+1}}{n_i + k} \]

is either an integer or a half-integer. Moreover, several related conjectures are proposed.

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1 Introduction

Shapiro [13] introduced the Catalan triangle \((B_{n,k})_{n\geq k\geq 0}\), where \(B_{n,k} := \frac{k}{n} \binom{2n}{n-k}\) (1 \(\leq k \leq n\)) and proved that \(\sum_{k=1}^{n} B_{n,k} = \frac{1}{2} \binom{2n}{n}\), i.e.,

\[ \sum_{k=1}^{n} k \binom{2n}{n-k} = \frac{n}{2} \binom{2n}{n}. \] (1.1)

Recently, Gutiérrez et al. [9], Miana and Romero [11], and Chen and Chu [5] studied the binomial sums \(\sum_{k=1}^{n} k^m \binom{2n}{n-k}^2\). On the other hand, Miana and Romero [11] have proved the following identity:

\[ \sum_{p=1}^{i} B_{n,k}B_{n,n+k-i}(n + 2k - i)^3 = \binom{2n}{n} \binom{2n-2}{i-1}(n^2 + 4n - 2ni + i^2), \] (1.2)
and asked: Are there polynomials \( P_m(n, i) \) and \( Q_m(n, i) \) of integral coefficients such that

\[
\sum_{k=1}^{i} B_{n,k} B_{n,n-k-1}(n+2p-i)^{2m+1} = \binom{2n}{n} \left( \frac{2n-2}{i-1} \right) \frac{P_m(n, i)}{Q_m(n, i)}
\]

for \( m \in \mathbb{N} \) and \( 1 \leq i \leq n \)?

Our first aim is to give a positive answer to their question. To this end, consider the sum

\[
\Theta_{2m+1}(n, r) := \sum_{\ell=1}^{n-r} \ell^m (\ell + r)^m (2\ell + r) \left( \binom{2n}{n-\ell} \binom{2n}{n+\ell+r} \right),
\]

and define the \( \alpha \)-coefficient by

\[
\alpha_k(m, n, r) := \sum_{i=0}^{k} \left( \binom{2n+r}{i} \binom{2k-2n-r}{k-i} \right) \frac{(n-i)^2 + r(n-i)}{(n+r-2k)_{2k+1}},
\]

where \( (a)_n = a(a+1) \cdots (a+n-1) \) is the Pochhammer symbol.

**Theorem 1.1.** For \( m, n, r \geq 0 \), there holds

\[
\Theta_{2m+1}(n, r) = n \binom{2n}{n} \frac{2n}{n-r-1} \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+r+1}{k+1} k!(k+1)! \frac{1}{2n-k} \alpha_k(m, n, r).
\]

**Remark.** When \( r = 0 \) Theorem 1.1 reduces to Theorem 3 in [5]. Writing \( (2k + r)^{2m} = (4k(k+r)+r^2)^m = \sum_{i=0}^{m} \binom{m}{i} 4^i (k+r)^i r^{2m-2i} \), we see from (1.4) that

\[
\sum_{k=1}^{n-r} B_{n,k} B_{n,k+r}(2k+r)^{2m+1} = \sum_{i=0}^{m} \binom{m}{i} 4^i \Theta_{2i+1}(n, r) r^{2m-2i},
\]

which gives a positive answer to the question (1.3) by Theorem 1.1. By the way, according to [5, (13g)] we have

\[
\sum_{p=1}^{n} p^9 (B_{n,p})^2 = \frac{n}{2n-5} (30n^5 - 150n^4 + 252n^3 - 185n^2 + 65n - 9).
\]

So the answer to question (2) in [11] is negative.

Secondly, motivated by the divisibility results in [2, 4, 5, 7, 9, 11, 13, 16], we shall prove the following theorems.

**Theorem 1.2.** For all nonnegative integers \( n \) and \( r \geq 1 \),

\[
n^{-2} \binom{2n}{n}^{-1} \sum_{k=1}^{n} \binom{2n}{n-k} k^{2r+1}
\]

is always a half-integer.
Theorem 1.3. For all positive integers \( n_1, \ldots, n_m \) and any nonnegative integer \( r \),
\[
S_{2r+1}(n_1, \ldots, n_m) := n_1^{-1} \binom{n_1 + n_m}{n_1}^{-1} \sum_{k=1}^{n_1} k^{2r+1} \prod_{i=1}^{m} \binom{n_i + n_i + i}{n_i} (n_{m+1} = n_1)
\]
is either an integer or a half-integer.

For example, we have
\[
S_7(3, 3, 2, 3) = \frac{10233}{2}, \quad S_{11}(3, 3, 2, 2) = 2448.
\]

Theorem 1.4. Let \( n \) be a prime power. Let \( r \geq 0 \) and \( s \geq 1 \) such that \( r \not\equiv s \pmod{2} \).

Then
\[
\binom{2n}{n}^{-1} \sum_{k=1}^{n} k^r B_{n,k}^s
\]
is either an integer or a half-integer.

Remark. It seems that Theorem 1.4 holds for any positive integer \( n \). For example, one can easily check that
\[
\sum_{k=1}^{6} k^r B_{6,k}^s = 132^s + 2^r165^s + 3^r110^s + 4^r44^s + 5^r10^s + 6^r
\]
is divisible by \( \frac{1}{2} \binom{12}{6} = 2 \times 3 \times 7 \times 11 \).

The rest of this paper is organized as follows. We prove Theorems 1.1, 1.2, 1.3 and 1.4 in Sections 2, 3, 4 and 5, respectively. Some consequences of Theorem 1.2 are given in Section 6. Some open problems and conjectures are proposed in Section 7.

2 Proof of Theorem 1.1

Let \( x_0, x_1, \ldots, x_m \) be different values of a variable \( x \). Then the Newton interpolation formula (see [12, Chapter 1]) implies that
\[
x^m = \sum_{k=0}^{m} \left( \sum_{i=0}^{k} \frac{x_i^m}{\prod_{j=0; j \neq i}^{k} (x_i - x_j)} \right) \prod_{i=0}^{k-1} (x - x_i).
\]

Letting \( x_i = (n - i)^2 + r(n - i) \) in (2.1), we have
\[
x^m = \sum_{k=0}^{m} \alpha_k(m, n, r) \prod_{i=0}^{k-1} ((n - i)^2 + r(n - i) - x),
\]

(2.2)
where $\alpha_k(m, n, r)$ is given by (1.5).

Replacing $x$ by $x(x + r)$ in (2.2) and noticing that $(n - i)^2 + r(n - i) + x(x + r) = (n - x - i)(n + x + r - i)$, we get

$$x^{m}(x + r)^{m} = \sum_{k=0}^{m} \binom{n-x}{k} \binom{n+x+r}{k} k!^2 \alpha_k(m, n, r). \quad (2.3)$$

Using (2.3) to rewrite $\ell^n(\ell + r)^{m}$ and applying the binomial relation

$$\binom{n - \ell}{k} \binom{n + \ell + r}{k} \binom{2n}{n - \ell} \binom{2n}{n + \ell + r} = \binom{2n}{k} \binom{2n - k}{n - \ell} \binom{2n - k}{n + \ell} \binom{2n - k}{n - \ell - r}. \quad (2.4)$$

one has

$$\Theta_{2m+1}(n, r) = \sum_{\ell=1}^{n-r} (2\ell + r) \binom{2n}{n - \ell} \binom{2n}{n + \ell + r} \sum_{k=0}^{m} \binom{n - \ell}{k} \binom{n + \ell + r}{k} \alpha_k(m, n, r)$$

$$= \sum_{k=0}^{m} \frac{2n}{k} \alpha_k(m, n, r) \sum_{\ell=1}^{n-r} \binom{2\ell - k}{n - \ell} \binom{2n - k}{n + \ell} \binom{2n - k}{n - \ell - r}. \quad (2.4)$$

Noticing that

$$\binom{x}{n + \ell} \binom{x}{n - \ell - r} = \binom{n + \ell}{n + \ell} \binom{x}{n + \ell + 1} \binom{x}{n - \ell - 1},$$

we have

$$\sum_{\ell=1}^{n-r} \binom{2\ell - k}{n - \ell} \binom{2n - k}{n + \ell} = (n + 1) \binom{2n - k}{n + 1} \binom{2n - k}{n - r - 1}. \quad (2.4)$$

The theorem then follows by observing that

$$(n + 1) \binom{2n}{k} \binom{2n - k}{n + 1} \binom{2n - k - 1}{n - r - 1} = n \binom{2n}{n} \binom{2n}{n - r - 1} \binom{2n - r + 1}{k + 1} \binom{2n - k}{k + 1}.$$

For the reader’s convenience, we derive the first values of $\Theta_{2m+1}(n, r)$ from Theorem 1.1.

$$\Theta_1(n, r) = \binom{n}{n} \binom{2n}{n - r - 1} n^3,$$

$$\Theta_3(n, r) = \binom{2n}{n} \binom{2n - 2}{n - r - 1} \frac{n^3(3n^3 - 5n - r^2 + 1)}{2n - 3},$$

$$\Theta_5(n, r) = \binom{2n}{n} \binom{2n - 2}{n - r - 1} \frac{n^3(6n^3 - 12n^2 - 4nr^2 + 6n + r^2 - 1)}{2n - 3},$$

$$\Theta_7(n, r) = \binom{2n}{n} \binom{2n - 2}{n - r - 1} n^3 \left\{ \frac{30n^5 - 150n^4 + 252n^3 - 30n^3r^2 + 91n^2r^2 - 185n^2}{(2n - 3)(2n - 5)} \right\}.$$
Therefore, by (1.6) we have
\[ \sum_{k=1}^{n-r} B_{n,k} B_{n,k+r} (2k+r)^3 = \binom{2n}{n} \binom{2n-2}{n-r-1} (4n+r^2), \quad (2.5) \]
\[ \sum_{k=1}^{n-r} B_{n,k} B_{n,k+r} (2k+r)^5 = \binom{2n}{n} \binom{2n-2}{n-r-1} \left( \frac{16n(3n^2-5n^2+1)}{2n-3} + 8nr^2 + r^4 \right), \]
\[ \sum_{k=1}^{n-r} B_{n,k} B_{n,k+r} (2k+r)^7 = \binom{2n}{n} \binom{2n-2}{n-r-1} \times \left( \frac{64n(6n^3-12n^2-4nr^2+6n+r^2-1)+48n(3n^2-5n-r^2+1)r^2}{2n-3} + 12nr^4 + r^6 \right). \]

Eq. (2.5) is an equivalent form of [11, Theorem 2.3], i.e., (1.2).

3 Proof of Theorem 1.2

Proof. As in (2.2) we can prove the following identity (see [5, (9)]) by the Newton interpolation formula:
\[ \ell^{2r} = \sum_{k=0}^{r} \binom{n-\ell}{k} \binom{n+\ell}{k} \frac{2k!}{(2n-2k)_{2k+1}} \sum_{i=0}^{k} \binom{2n}{i} \binom{2k-2n}{k-i} (n-i)^{2r+1}. \quad (3.1) \]

It follows from (3.1) and (1.1) that
\[ \sum_{\ell=1}^{n} \binom{2n}{n-\ell} \ell^{2r+1} = \sum_{\ell=1}^{n} \sum_{k=0}^{r} \binom{2n-2k}{n-k-\ell} \frac{\ell}{n-k} \sum_{i=0}^{k} \binom{2n}{i} \binom{2k-2n}{k-i} (n-i)^{2r+1} \]
\[ = \frac{1}{2} \sum_{k=0}^{r} f_{n,k}(r), \quad (3.2) \]

where
\[ f_{n,k}(r) := \binom{2n-2k}{n-k} \sum_{i=0}^{k} \binom{2n}{i} \binom{2k-2n}{k-i} (n-i)^{2r+1}. \]

We now show by induction on \( r \) that \( f_{n,k}(r) \) is divisible by \( n_{\min(2,r+1)}(2n) \). For \( r = 0 \), writing \( f_{n,k}(0) \) as
\[ 2n \left( \binom{2n-2k}{n-k} \sum_{i=0}^{k} \binom{2n-1}{i} \binom{2k-2n}{k-i} - n \binom{2n-2k}{n-k} \sum_{i=0}^{k} \binom{2n}{i} \binom{2k-2n}{k-i} \right), \]
we see, by the Chu-Vandermonde formula, that
\[ f_{n,k}(0) = n \binom{2n - 2k}{n - k} \left( \binom{2k - 1}{k} - \binom{2k}{k} \right) = \begin{cases} 0, & \text{if } k > 0, \\ \binom{2n}{n}, & \text{if } k = 0. \end{cases} \]

Thus, for \( r = 0 \) we are done. For \( r \geq 1 \), suppose that \( f_{n,k}(r-1) \) is divisible by \( n^{\min(2r, n)} \) for all \( n, k \). Applying the relations
\[(n - i)^{2r+1} = (n - i)^{2r-1}(n^2 - (2n - i)i) \text{ and } \binom{2n}{i}(2n - i)i = 2n(2n - 1)\binom{2n - 2}{i - 1},\]
we have
\[ f_{n,k}(r) = n^2 f_{n,k}(r-1) - 2n(2n - 1)f_{n-1,k-1}(r - 1). \tag{3.3} \]

By the induction hypothesis, \( n^2 f_{n,k}(r-1) \) is divisible by \( n^2 \binom{2n}{n} \) and \( 2n(2n - 1)f_{n-1,k-1}(r - 1) \) is divisible by
\[ 2n(2n - 1)\binom{2n - 2}{n - 1} = n^2 \binom{2n}{n}. \]

This proves that \( f_{n,k}(r) \) is also divisible by \( n^2 \binom{2n}{n} \) for \( r \geq 1 \). Finally, it is also easy to check that
\[ n^{-2} \binom{2n}{n}^{-1} \sum_{k=0}^{r} f_{n,k}(r) \]
is an odd integer for \( r \geq 1 \) by using (3.3) and induction on \( r \). The details are left to the reader.

Here are some examples for small \( r \):
\[
\begin{align*}
\sum_{k=1}^{n} \binom{2n}{n - k} k^3 &= \frac{n^2}{2} \binom{2n}{n}, \\
\sum_{k=1}^{n} \binom{2n}{n - k} k^5 &= \frac{n^2}{2} \binom{2n}{n} (2n - 1), \\
\sum_{k=1}^{n} \binom{2n}{n - k} k^7 &= \frac{n^2}{2} \binom{2n}{n} (6n^2 - 8n + 3), \\
\sum_{k=1}^{n} \binom{2n}{n - k} k^9 &= \frac{n^2}{2} \binom{2n}{n} (24n^3 - 60n^2 + 54n - 17).
\end{align*}
\]

Remark. Note that Shapiro, Woan and Getu [14] proved that
\[
\sum_{n=1}^{\infty} \binom{n}{k} B_{n,k} x^n = \sum_{s=1}^{\lfloor n/2 \rfloor} \frac{m(r, s)x^s}{(1 - 4x)^{(s+1)/2}}, \tag{3.4}
\]
where \( m(r, s) \) denotes the number of permutations of \( \{1, \ldots, r\} \) with \( s \) runs and \( s \) slides. Using the binomial theorem, it is easy to give a formula for \( \sum_{k=1}^{n} k^r B_{n,k} \) involving \( m(r, s) \) from (3.4). A natural question is whether we can deduce Theorem 1.2 from (3.4).
4 Proof of Theorem 1.3

We will need the Pfaff-Saalshutz identity (see [1, p. 69] or [3, p. 43, (A)]) :

\[
\binom{n_1 + 2}{n_1 + k} \binom{n_2 + 3}{n_2 + k} \binom{n_3 + n_1}{n_3 + k} = \sum_{s=0}^{n_1-k} \frac{(n_1 + 2 + n_3 - k - s)!}{s!(s+2k)!(n_1 - k - s)!(n_2 - k - s)!(n_3 - k - s)!},
\]

(4.1)

where \( \frac{1}{n_1} = 0 \) if \( n_1 < 0 \).

For any positive integers \( a_1, \ldots, a_l \), let

\[
C(a_1, \ldots, a_l; k) = \prod_{i=1}^{l} \binom{a_i + a_{i+1}}{a_i + k},
\]

where \( a_{l+1} = a_1 \). Then

\[
S_{2r+1}(n_1, \ldots, n_m) = \frac{(n_1 - 1)!n_m!}{(n_1 + n_m)!} \sum_{k=1}^{n_1} C(n_1, \ldots, n_m; k)k^{2r+1}.
\]

(4.2)

Observe that for \( m \geq 3 \), we have

\[
C(n_1, \ldots, n_m; k) = \frac{(n_2 + 3)!(n_m + n_1)!}{(n_1 + n_2)!(n_m + n_3)!} \binom{n_1 + 2}{n_1 + k} \binom{n_2 + 3}{n_2 + k} C(n_3, \ldots, n_m; k),
\]

and, by letting \( n_3 \to \infty \) in (4.1),

\[
\binom{n_1 + 2}{n_1 + k} \binom{n_1 + 2}{n_2 + k} = \sum_{s=0}^{n_1-k} \frac{(n_1 + 2)!}{s!(s+2k)!(n_1 - k - s)!(n_2 - k - s)!}.
\]

Plugging these into (4.2) we can write its right-hand side as

\[
S_{2r+1}(n_1, \ldots, n_m) = \frac{(n_2 + 3)!(n_1 - 1)!n_m!}{(n_1 + n_3)!} \sum_{k=1}^{n_1} \sum_{s=0}^{n_1-k} \frac{C(n_3, \ldots, n_m; k)k^{2r+1}}{s!(s+2k)!(n_1 - k - s)!(n_2 - k - s)!}.
\]

\[
= \frac{(n_2 + 3)!(n_1 - 1)!n_m!}{(n_1 + n_3)!} \sum_{k=1}^{n_1} \sum_{l=1}^{l} \frac{C(n_3, \ldots, n_m; k)k^{2r+1}}{(l - k)!(l + k)!(n_1 - l)!(n_2 - l)!},
\]

where \( l = s + k \). Now, in the last sum making the substitution

\[
\frac{C(n_3, \ldots, n_m; k)}{(l - k)!(l + k)!} = \frac{(n_m + n_3)!}{(n_3 + l)!(n_m + l)!} C(l, n_3, \ldots, n_m; k),
\]

we obtain the following recurrence relation

\[
S_{2r+1}(n_1, \ldots, n_m) = \sum_{l=1}^{n_1} \binom{n_1 - 1}{l - 1} \binom{n_2 + n_3}{n_2 - l} S_{2r+1}(l, n_3, \ldots, n_m).
\]

(4.3)

By induction on \( m \) and using Theorem 1.2, we complete the proof.

By iteration of (4.3) for \( r = 0, 1 \), we obtain the following result.
Corollary 4.1. For \( m \geq 3 \) and all positive integers \( n_1, \ldots, n_m \), there hold

\[
\begin{align*}
\sum_{k=1}^{n_1} k \prod_{i=1}^{m} \left( n_i + k \right) &= \frac{n_1}{2} \left( n_1 + n_m \right) \sum_{\lambda} \left( \lambda_{m-2} + n_m - 1 \right) \prod_{i=1}^{m-2} \left( \frac{\lambda_{i-1} - 1}{\lambda_i - 1} \right) \left( n_{i+1} + n_{i+2} \right), \\
\sum_{k=1}^{n_1} k^3 \prod_{i=1}^{m} \left( n_i + k \right) &= \frac{n_1 n_m}{2} \left( n_1 + n_m \right) \sum_{\lambda} \left( \lambda_{m-2} + n_m - 2 \right) \prod_{i=1}^{m-2} \left( \frac{\lambda_{i-1} - 1}{\lambda_i - 1} \right) \left( n_{i+1} + n_{i+2} \right),
\end{align*}
\]

where \( n_{m+1} = \lambda_0 = n_1 \) and the sums are over all sequences \( \lambda = (\lambda_1, \ldots, \lambda_{m-2}) \) of positive integers such that \( n_1 \geq \lambda_1 \geq \cdots \geq \lambda_{m-2} \).

Note that the following identity was established in [7]:

\[
\sum_{k=-n_1}^{n_1} (-1)^k \prod_{i=1}^{m} \left( n_i + k \right) = \left( \frac{n_1 + n_m}{n_1} \right) \sum_{n_1 \geq \lambda_1 \geq \cdots \geq \lambda_{m-2} \geq 0} \prod_{i=1}^{m-2} \left( \frac{\lambda_{i-1} - 1}{\lambda_i - 1} \right) \left( n_{i+1} + n_{i+2} \right),
\]

where \( n_{m+1} = \lambda_0 = n_1 \).

5 Proof of Theorem 1.4

We need the following theorem of Lucas (see, for example, [6]) and [15] for a recent application.

Theorem 5.1 (Lucas’ theorem). Let \( p \) be a prime, and let \( a_0, b_0, \ldots, a_m, b_m \in \{0, \ldots, p-1\} \). Then

\[
\left( \frac{\sum_{i=0}^{m} a_i p^i}{\sum_{i=0}^{m} b_i p^i} \right) \equiv \prod_{i=0}^{m} \left( \frac{a_i}{b_i} \right) \pmod{p}.
\]

Let \( r + s \equiv 1 \pmod{2} \) and \( s \geq 1 \). Setting \( n_1 = \cdots = n_m = n \) in Theorem 1.3, one sees that

\[
\sum_{k=1}^{n} k^{r+s} \binom{2n}{n-k}^s
\]

is divisible by \( \binom{2n-1}{n} \). Note that \( k \binom{2n}{n-k} = n B_{n,k} \) is clearly divisible by \( n \). Therefore,

\[
\sum_{k=1}^{n} k^r B_{n,k}^s = n^{-s} \sum_{k=1}^{n} k^{r+s} \binom{2n}{n-k}^s
\]

is divisible by

\[
\binom{2n-1}{n}/\gcd\left(\binom{2n-1}{n}, n^s\right).
\]
Now suppose that \( n = p^\alpha \) is a prime power. It follows immediately from Lucas’ theorem that
\[
\left( \begin{array}{c}
2^{\alpha+1} - 1 \\
2^\alpha
\end{array} \right) = \left( \sum_{k=0}^{\alpha} 2^k \right) \equiv 1 \pmod{2},
\]
\[
2 \left( \begin{array}{c}
2p^\alpha - 1 \\
p^\alpha
\end{array} \right) \equiv 2 \pmod{p} \quad (p > 2).
\]
Namely, we always have
\[
\left( \begin{array}{c}
2n - 1 \\
n
\end{array} \right) \equiv 1 \pmod{n},
\]
and thereby
\[
\gcd \left( \left( \begin{array}{c}
2n - 1 \\
n
\end{array} \right), n^a \right) = 1.
\]
(5.1)
This completes the proof.

Remark. Besides prime powers, there are some other natural numbers satisfying (5.1). For example, if \( p \) and \( p^2 + p + 1 \) are both odd primes, then it is easy to see from Lucas’ theorem that \( n = p(p^2 + p + 1) \) satisfies (5.1). In fact, the following are all the natural numbers \( n \) less than 500 not being prime powers but satisfying (5.1):
\[
\]
Therefore, Theorem 1.4 is also true for these numbers.

6 Consequences of Theorem 1.3

Letting \( n_{2i-1} = m \) and \( n_{2i} = n \) for \( 1 \leq i \leq r \) in Theorem 1.3 and noticing the symmetry of \( m \) and \( n \), we obtain

Corollary 6.1. For all positive integers \( m, n, r \) and any nonnegative integer \( a \),
\[
2 \sum_{k=1}^{m} k^{2a+1} \left( \begin{array}{c}
m + n \\
m + k
\end{array} \right)^r \left( \begin{array}{c}
m + n \\
(n + k)
\end{array} \right)^r
\]
is divisible by \( \frac{mn}{\gcd(m,n)} \left( \begin{array}{c}
m + n \\
m
\end{array} \right) \).

Letting \( n_{3i-2} = l, n_{3i-1} = m \) and \( n_{3i} = n \) for \( 1 \leq i \leq r \) in Theorem 1.3, we obtain

Corollary 6.2. For all positive integers \( l, m, n, r \) and any nonnegative integer \( a \),
\[
2 \sum_{k=1}^{l} k^{2a+1} \left( \begin{array}{c}
l + m \\
l + k
\end{array} \right)^r \left( \begin{array}{c}
l + m \\
(n + l)
\end{array} \right)^r \left( \begin{array}{c}
l + m \\
(n + k)
\end{array} \right)^r
\]
is divisible by \( \frac{lm}{\gcd(l,m)} \left( \begin{array}{c}
l + m \\
l
\end{array} \right), \frac{mn}{\gcd(m,n)} \left( \begin{array}{c}
m + n \\
m
\end{array} \right) \) and \( \frac{nl}{\gcd(n,l)} \left( \begin{array}{c}
l + l \\
l
\end{array} \right) \).
Letting $m = 2r + s$, $n_1 = n_3 = \cdots = n_{2r-1} = n + 1$ and letting all the other $n_i$ be $n$ in Theorem 1.3, we get

**Corollary 6.3.** For all positive integers $r$, $s$, $n$ and any nonnegative integer $a$,

$$\sum_{k=1}^{n} k^{2a+1} \binom{2n+1}{n+k+1} \binom{2n+1}{n+k}^r \binom{2n}{n+k}^s$$

is divisible by $\frac{n(n+1)(2n+1)}{2n} \binom{2n+1}{n}$.

Clearly Theorem 1.3 can be restated in the following form.

**Theorem 6.4.** For all positive integers $n_1, \ldots, n_m$ and any nonnegative integer $r$,

$$2(n_1 - 1)! \prod_{i=1}^{m} \frac{(n_i + n_{i+1})!}{(2n_i)!} \sum_{k=1}^{n_1} k^{2r+1} \prod_{i=1}^{m} \binom{2n_i}{n_i + k},$$

where $n_{m+1} = 0$, is an integer.

It is not hard to see that, for all positive integers $m$ and $n$, the expression $\frac{(2m)! (2n)!}{2(m+n)! m! n!}$ is an integer by considering the power of a prime dividing a factorial. Letting $n_{r} = m$ and $n_{r+1} = \cdots = n_{r+s} = n$ in Theorem 6.4 and noticing the symmetry of $m$ and $n$, we obtain

**Corollary 6.5.** For all positive integers $m$, $n$, $r$, $s$ and any nonnegative integer $a$,

$$\sum_{k=1}^{m} k^{2a+1} \binom{2m}{m+k}^r \binom{2n}{n+k}^s$$

is divisible by $\frac{(2m)! (2n)!}{2(m+n)! (m-1)! (n-1)! \gcd(m,n)}$.

In particular, we find that

$$\sum_{k=1}^{n} k^{2a+1} \binom{4n}{2n+k}^r \binom{2n}{n+k}^s$$

is divisible by $n \binom{4n}{n}$, and

$$\sum_{k=1}^{n} k^{2a+1} \binom{6n}{3n+k}^r \binom{2n}{n+k}^s$$

is divisible by $\frac{(6n)! (2n-1)!}{(4n)! (3n-1)! (n-1)!}$.

Using Lucas’ theorem, similarly to Theorem 1.4, we can deduce the following result immediately.
Corollary 6.6. Let $n$ be a power of 2. Let $r \geq 0$ and $s, t \geq 1$ such that $r + s + t \equiv 1 \pmod{2}$. Then
\[ \sum_{k=1}^{n} k^{r} B_{2n,k}^{s} B_{n,k}^{t} \]
is divisible by $\binom{4n - 1}{n - 1}$.

From Theorem 6.4 it is easy to see that
\[ 2(n_1 - 1)! \prod_{i=1}^{m} \frac{(n_i + n_{i+1})!}{(2n_i)!} \sum_{k=1}^{n_{2a+1}} \prod_{i=1}^{m} \binom{2n_i}{n_i + k}^{r_i}, \]
where $n_{m+1} = 0$, is a nonnegative integer for all $r_1, \ldots, r_m \geq 1$. For $m = 3$, letting $(n_1, n_2, n_3)$ be $(n, 3n, 2n)$, $(2n, n, 3n)$, or $(2n, n, 4n)$, and noticing the symmetry of $n_1$ and $n_3$, we obtain the following two corollaries.

Corollary 6.7. For all positive integers $r, s, t, n$ and any nonnegative integer $a$,
\[ \sum_{k=1}^{n} k^{2a+1} \left( \begin{array}{c} 6n \\ 3n + k \end{array} \right)^{r} \left( \begin{array}{c} 4n \\ 2n + k \end{array} \right)^{s} \left( \begin{array}{c} 2n \\ n + k \end{array} \right)^{t} \]
is divisible by both $n\binom{6n}{n}$ and $3n\binom{6n}{3n}$.

Corollary 6.8. For all positive integers $r, s, t, n$ and any nonnegative integer $a$,
\[ \sum_{k=1}^{n} k^{2a+1} \left( \begin{array}{c} 8n \\ 4n + k \end{array} \right)^{r} \left( \begin{array}{c} 4n \\ 2n + k \end{array} \right)^{s} \left( \begin{array}{c} 2n \\ n + k \end{array} \right)^{t} \]
is divisible by $2n\binom{8n}{3n}$.

7 Concluding remarks and open problems

It is easy to prove that
\[ \sum_{k=1}^{n} \binom{2n}{n - k} k^2 = 4^{n-1} n \]
(see [14]). Furthermore, similarly to (3.2), there holds
\[ \sum_{k=1}^{n} \binom{2n}{n - k} k^{2r} = \sum_{k=0}^{r-1} 4^{n-k-1} \sum_{i=0}^{k} \binom{2n}{i} \binom{2k - 2n}{k - i} (n - i)^{2r-1}, \quad r \geq 1. \quad (7.1) \]
In particular, for \( r = 2, 3, 4, 5 \), we have

\[
\sum_{k=1}^{n} \binom{2n}{n-k} k^4 = 2^{2n-3}n(3n - 1),
\]

\[
\sum_{k=1}^{n} \binom{2n}{n-k} k^6 = 2^{2n-4}n(15n^2 - 15n + 4),
\]

\[
\sum_{k=1}^{n} \binom{2n}{n-k} k^8 = 2^{2n-5}n(105n^3 - 210n^2 + 147n - 34),
\]

\[
\sum_{k=1}^{n} \binom{2n}{n-k} k^{10} = 2^{2n-6}n(945n^4 - 3150n^3 + 4095n^2 - 2370n + 496).
\]

Let \( \alpha(N) \) denote the number of 1’s in the binary expansion of \( N \). For example, \( \alpha(255) = 8 \) and \( \alpha(256) = 1 \).

**Conjecture 7.1.** Let \( n, r \geq 1 \). Then

\[
\sum_{k=1}^{n} \binom{2n}{n-k} k^{2r}
\]

is divisible by \( 2^{2n-\min\{\alpha(n), \alpha(r)\}-1} \).

Note that it is clear from (7.1) that (7.2) is divisible by \( 4^{n-r} \), assuming that \( n \geq r \). The statistic \( \alpha(n) \) appears in Conjecture 7.1 because the divisibility of (7.2) also depends on \( n \).

**Conjecture 7.2.** Let \( n, r \geq 1 \). Then

\[
\sum_{k=1}^{n} B_{n,k}^{2r+1} \equiv \binom{2n-1}{n} \mod \binom{2n}{n}
\]

if and only if \( n = 2^a - 2^b \) for some \( 0 \leq b < a \).

For example, we have

\[
\sum_{k=1}^{7} B_{7,k}^{3} = 354331692 \equiv 1716 \pmod{3432},
\]

\[
\sum_{k=1}^{12} B_{12,k}^{3} = 96906387191038334 \equiv 1352078 \pmod{2704156},
\]

\[
\sum_{k=1}^{13} B_{13,k}^{3} = 5066711735118128200 \equiv 0 \pmod{10400600},
\]

\[
\sum_{k=1}^{16} B_{16,k}^{3} = 786729115199980286001225 \equiv 0 \pmod{601080390}.
\]
Conjecture 7.3. Let \( r \geq 0 \) and \( s \geq 1 \). Then for \( n \geq 4^s - 1 \),
\[
\sum_{k=1}^{n} k^{2r+1} B_{n,k}^{2s} \equiv \begin{cases} 
\binom{2n-1}{n} 4^{s-1}, & \text{if } n = 4^s - 1 \text{ or } n = 2^a + 1, \\
0, & \text{otherwise}.
\end{cases} \mod \left( \frac{2n}{n} \right) 4^{s-1}.
\]

Conjecture 7.4. Let \( n, r, s \geq 1 \). Then
\[
\sum_{k=1}^{n} k^{2r} B_{n,k}^{2s+1} \equiv \begin{cases} 
\binom{2n-1}{n}, & \text{if } n = 2^a - 1, \\
0, & \text{otherwise}.
\end{cases} \mod \left( \frac{2n}{n} \right).
\]

The following statement is a generalization of Corollary 6.6.

Conjecture 7.5. Let \( m, n, s, t \geq 1 \), and \( r \geq 0 \) such that \( r + s + t \equiv 1 \pmod{2} \). Then
\[
\sum_{k=1}^{n} k^{r} B_{m,k}^{s} B_{n,k}^{t} \]

is divisible by \( \frac{1}{2} \frac{(2m)!/(2n)!}{\min\{m+n\}} \).

Furthermore, for some special cases, we have

Conjecture 7.6. Let \( n, r, s \geq 1 \) such that \( r \not\equiv s \pmod{2} \). Then
\[
\sum_{k=1}^{n} B_{r,n,k}^{r} B_{s,2n,k}^{s} \equiv \begin{cases} 
\binom{4n}{n} 2^{\min\{r,s\}-2}, & \text{if } n = \frac{2^a(2b+1)+1}{3}, \\
0, & \text{otherwise}.
\end{cases} \mod \left( \frac{4n}{n} \right) 2^{\min\{r,s\}-1}.
\]

Conjecture 7.7. Let \( n, r, s \geq 1 \) such that \( r \not\equiv s \pmod{2} \). Then
\[
\sum_{k=1}^{n} B_{r,n,k}^{r} B_{s,n+1,k}^{s} \equiv \begin{cases} 
\binom{2n}{n} 2^{\min\{r,s\}-1}, & \text{if } \begin{cases} 
 r < s, & n = 2^a - 2^b \ (b \geq 1) \\
 r > s, & n = 2^a - 1
\end{cases} \\
0, & \text{otherwise}.
\end{cases} \mod \left( \frac{2n}{n} \right) 2^{\min\{r,s\}}.
\]

The following two conjectures are refinements of Corollaries 6.7 and 6.8 for \( 2a + 1 = r + s + t \).

Conjecture 7.8. For any positive integers \( r, s, t \) such that \( r + s + t \equiv 1 \pmod{2} \),
\[
\sum_{k=1}^{n} B_{r,n,k}^{r} B_{s,2n,k}^{s} B_{3n,k}^{t} \]

is divisible by both \( \frac{1}{3} \binom{6n}{n} \) and \( \binom{6n}{3n} \).
Conjecture 7.9. For any positive integers $r, s, t$ such that $r + s + t \equiv 1 \pmod{2}$,
\[
\sum_{k=1}^{n} B_{n,k}^r B_{2n,k}^s B_{4n,k}^t
\]
is divisible by $\binom{8n}{3n}$.

Finally it is natural to introduce the $q$-Catalan triangles with entries given by
\[
B_{n,k}(q) := \frac{1 - q^k}{1 - q^n} \binom{2n}{n-k}, \quad 1 \leq k \leq n,
\]
where the $q$-binomial coefficient is defined by
\[
\left[\begin{array}{c}
M \\
N
\end{array}\right] := \begin{cases}
\prod_{i=1}^{N} \frac{1 - q^{M-N+i}}{1 - q^i}, & \text{if } 0 \leq N \leq M, \\
0, & \text{otherwise}.
\end{cases}
\]

Let $\Phi_n(x)$ be the $n$-th cyclotomic polynomial. It is not difficult to see from [10, Eq. (10)] or [8, Proposition 2.2] that
\[
B_{n,k}(q) = \left(\prod_i \Phi_i(q)\right) \left(\prod_d \Phi_d(q)\right),
\]
where the first product is over all positive integers $i$ such that $i \mid k$ and $i \nmid n$ and the second product is over all positive integers $d$ such that $\lfloor (n-k)/d \rfloor + \lfloor (n+k)/d \rfloor < \lfloor 2n/d \rfloor$ and $d$ does not divide $n$.

Now, the obvious identity
\[
\frac{1 - q^k}{1 - q^n} \binom{2n}{n-k} q^{\binom{k}{2}} = \binom{2n-1}{n-k} q^{\binom{k}{2}} - \binom{2n-1}{n-k-1} q^{\binom{k+1}{2}}
\]
implies the following $q$-analogue of (1.1):
\[
\sum_{k=1}^{n} \frac{1 - q^k}{1 - q^n} \binom{2n}{n-k} q^{\binom{k}{2}} = \frac{1}{1 + q^n} \binom{2n}{n}.
\]

In view of the identity (7.3) and the results in [7], it would be interesting to find a $q$-analogue of Theorems 1.2 and 1.3.

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